CHAPTER 6

algebraic EXPRESSIONs iii

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6.1 CONCEPT OF EXPANDING BRACKETS

Expand a single pair of brackets

$w(x + y)$ can be expanded to become $wx + wy$ as shown below.

$$w(x + y) = (w \times x) + (w \times y)$$
$$= wx + wy$$

RECALL

- $w \times x = wx$
- $w \times y = wy$
- wx and wy are unlike terms.
- unlike terms cannot be added together

Example 1

Expand the following algebraic expressions.

a) $r(s - 2t)$ b) $3e(2e - f + 4g)$

TIPS

- $(+r) \times (+s) = +rs$
- $(+r) \times (-s) = -rs$
- $(-r) \times (+s) = -rs$
- $(-r) \times (-s) = +rs$

Solution:

a) $r(s - 2t) = (r \times s) + [r \times (-2t)]$

b) $3e(2e - f + 4g)$

$= (3e \times 2e) + [3e \times (-f)] + (3e \times 4g)$

$= 6e^2 - 3ef + 12eg$

Exercise 6.1A

1. Expand the following algebraic expressions.

a) $p(q + r)$
b) $5(3x + y)$
c) $-w(x - y)$
d) $-3r(-t + 2s)$
e) $2q(2r - 3s + t)$
f) $\frac{1}{2}p(x + y)$

2. Expand the following expressions.

a) $-4m(-m - 3n)$
b) $4g(f/2 - 3g)$
c) $-n/2(2n + 8m)$
d) $4q/3(12p + 6q - 9r)$

Expand double pairs of brackets

(w + x)(y + z) can be expanded to
wy + wz + xy + xz as shown below.

(w + x)(y + z)

= (w x y) + (w x z) + (x x y) + (x x z)

= wy + wz + xy + xz

Example 2

Expand the following algebraic expressions.

a) $(2a + b)(c + 3d)$ b) $(m + 4)(m – 3)$

Solution:

a) $(2a + b)(c + 3d)$

$= (2a \times c) + (2a \times 3d) + (b \times c) + (b \times 3d)$

$= 2ac + 6ad + bc + 3bd$

b) $(m + 4)(m – 3)$

$= (m \times m) + [m \times (-3)] + (4 \times m) + [4 \times (-3)]$

$= m^2 – \underbrace{3m + 4m}_{\text{Like terms}} – 12$

$= m^2 + m - 12$

TIPS

- $(x \times w) = (w \times x) = wx$
- $(y \times w) = (w \times y) = wy$

Look at the products of $(a + b)$ and $(a - b)$.

$(a + b)(a - b)$

$= (a \times a) + [a \times (-b)] + (b \times a) + [b \times (-b)]$

$= a^2 - ab + ab + b^2 = a^2 - b^2$

Since $(a + b)(a - b) = (a - b)(a + b)$,

$(a - b)(a - b)$ is also equal to $a^2 - b^2$.

In general, $(a + b)(a - b) = (a - b)(a + b)$

$= a^2 - b^2$

Example 3

Expand the following expressions.

a) $(a + 2b)(a - 2b)$ b) $(2p + 5)(2p - 5)$

Solution:

a) $(a + 2b)(a - 2b)$

$= (a \times a) + [a \times (-2b)] + (2b \times a) + [2b \times (-2b)]$

$= a^2 - 2ab + 2ab - 4b^2$

$= a^2 - 4b^2$ ← $a^2 - 4b^2 = (a)^2 - (2b)^2$

b) $(2p + 5)(2p – 5)$

$= (2p \times 2p) + [2p \times (-5)] + (5 \times 2p) + [5 \times (-5)]$

$= 4p^2 – 10p + 10p – 25$

$= 4p^2 – 25$

$4p^2 – 25 = (2p)^2 – (5)^2$

NOTES

- -10p and 10p are like terms.
- $-10p + 10p = 0$

Thus, $(a + b)(a – b)$ can be written as $a^2 – b^2$ directly.

Example 4

Expand the following expressions.

a) $(2x + 7)(2x - 7)$ b) $(3k - 2h)(3k + 2h)$

Solution:

a) $(2x + 7)(2x - 7)$

$= (2x)^2 - (7)^2$

$= 4x^2 - 49$

b) $(3k - 2h)(3k + 2h)$

$= (3k)^2 - (2h)^2$

$= 9k^2 - 4h^2$

NOTES

$(a + b)(a - b) \neq a^2 + b^2$

Look at the expansions of $(a + b)^2$ and $(a – b)^2$.

$$(a + b)^2 = (a + b)(a + b)$$
$$= (a \times a) + (a \times b) + (b \times a) + (b \times b)$$
$$= a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2$$

TIPS

- ab and ba are like terms.
- $ab + ba = 2ab$
- $-ab – ba = -2ab$

$(a - b)^2$

$= (a - b)(a - b)$

$= (a \times a) + [a \times (-b)] + [(-b) \times a] + [(-b) \times (-b)]$

$= a^2 - ab - ba + b^2$

$= a^2 - 2ab + b^2$

Thus,

$(a + b)^2 = a^2 + 2ab + b^2$

and

$(a - b)^2 = a^2 - 2ab + b^2$

Example 5

Expand the following expressions.

a) $(2m + 3n)^2$ b) $(4s – t)^2$

Solution:

$(a + b)^2 = a^2 + 2ab + b^2$

a) $(2m + 3n)^2 = (2m)^2 + 2(2m)(3n) + (3n)^2$

$= 4m^2 + 12mn + 9n^2$

b) $(4s – t)^2 = (4s)^2 + 2(24s)(t) + (t)^2$

$= 16s^2 – 8st + t^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Exercise 6.1B

1. Expand the following expressions.

a) $(a + b)(c + 2d)$

b) $(2m – n)(2p – q)$

c) $(x + 7)(2y + 5)$

d) $(2p - 7)(q + 2)$

e) $(u – 2v)(3w + v)$

f) $(3k + 6)(k - 5)$

2. Expand the following expressions where each is the product of the sum and the difference of two terms.

a) $(x + 3y)(x – 3y)$

b) $(f + 5g)(f – 5g)$

c) $(v - 7w)(v + 7w)$

d) $(4h - 3k)(4h + 3k)$

e) $(5k + 2)(5k - 2)$

f) $(7 + a)(7 - a)$

3. Expand the following expressions.

a) $(m + n)^2$

b) $(c - d)$

c) $(3w + 4x)^2$

d) $(5t - u)^2$

e) $(-y + 2)^2$

f) $(-2p - 5q)^2$

6.2 CONCEPT OF FACTORIZATION OF ALGEBRAIC EXPRESSIONS

State the factors of an algebraic term

A number can be expressed as the product of its factors.

18 = 1 x 18
= 2 x 9
= 3 x 6

An algebraic term can be expressed as the product of its factors.

3ab = 1 x 3ab
= 3 x ab
= a x 3b
= b x 3a

State the common factors and the highest common factor of several algebraic terms

- ***Common factors of several Algebraic Terms***

Step 1: List all the factors of each algebraic term.

Step 2: Identify the factors which are common to all the terms.

Example 7

Find the common factors of the algebraic terms, 2ab and 4bc.

Solution:

2ab = 1 x 2ab

= 2 x ab

= a x 2b

= b x 2a

The factors of 2ab are **1**, **2**, a, **b**, 2a, **2b**, ab and 2ab.

RECALL

- The common factors are factors which are common to all the numbers.
- The HCF is the highest common factor for all the numbers.
- Examples:

- The factors of 8 are 1, 2, 4 and 8.
- The factors of 12 are 1, 2, 3, 4, 6 and 12.
- The common factors are 1, 2 and 4.
- The HCF is 4.

4bc = 1 x 4bc

= 2 x 2bc

= 4 x bc

= b x 4c

= c x 4b

= 2b x 2c

The factors of 4bc are **1**, **2**, 4, **b**, c, **2b**, 2c, 4b, 4c, bc, 2bc and 4bc.

Thus, the common factors of 2ab and 4bc are **1**, **2**, **b** and **2b**.

Highest Common Factor (HCF) of Several Algebraic Terms

Example 8

Find the highest common factor of the algebraic terms, 3uv and $6u^2$.

Solution:

3uv = 1 x 3uv
= 3 x uv
= u x 3v
= v x 3u

The factors of 3uv are **1**, **3**, **u**, v, **3u**, 3v, uv and 3uv.

$6u^2$ $= 1 \times 6u^2$

$= 2 \times 3u^2$

$= 3 \times 2u^2$

$= 6 \times u^2$

$= u \times 6u$

$= 2u \times 3u$

The factors of $6u^2$ are **1**, 2, **3**, 6, **u**, 2u, **3u**, 6u, u^2, $2u^2$, $3u^2$ and $6u^2$.

The common factors of 3uv and $6u^2$ are **1**, **3**, **u** and **3u**.

Thus, the highest common factor of 3uv and $6u^2$ is **3u**.

ANOTHER WAY

Using the division method to find the HCF.

- Divide the terms by a common factor until no more common factor exists.

3	$3uv, 6u^2$
u	$uv, 2u^2$
	$v, 2u$

The HCF of $3uv$ and $6u^2 = 3 \times u$

$= 3u$

This is a faster way of determining HCF where many algebraic terms are involved.

Exercise 6.2A

Find the highest common factor (HCF) of the following algebraic terms.

1. $2f$, $4g$
2. $4xy$, $8yz$
3. $5g^2$, $10gh$
4. n^2p, np^2
5. $8km^2$, $12mn$
6. ef^2, $2ef$, $4efg$
7. $6abc^2$, $12a^2bc$, $18ab^2c$

Factorize algebraic expressions

NOTES

Expansion

$$p(q + r) = pq + pr$$

Factorization

Factorization is the reverse process of expansion.

Factorize ab – ac

Consider an expression $2ab - 4ac$.

2	2ab, 4ac
a	ab, 2ac
	b, 2c

HCF = 2 x a

= 2a

NOTES

$$\frac{2ab - 4ac}{2a} = \frac{2ab}{2a} - \frac{4ac}{2a}$$

$$= b - 2c$$

2ab – 4ac = 2a(b – 2c)

HCF

Final quotient

Thus, 2ab – 4ac can be expressed as the product of 2a and (b – 2c).

Example 9

Factorize the following algebraic expressions.

a) $2pq + 4qr$ b) $mn - mn^2$

c) $3xy + 6yz - 9y^2$

Solution:

a) $2pq + 4qr = 2q(p + 2r)$

HCF: $2q$ — Final quotient: $(p + 2r)$

2	$2pq, 4qr$
q	$pq, 2qr$
	$p, 2r$

HCF = $2q$

Check:

$2q(p + 2r) = 2pq + 4qr$

$$b)\ mn - mn^2 = mn(1 - n)$$

HCF

Final quotient

Check:

$mn(1 - n) = mn - mn^2$

m	mn, mn^2
n	n, n^2
	$1, n$

HCF = mn

$$c)\ 3xy + 6yz - 9y^2 = 3y(x + 2z - 3y)$$

HCF

Final quotient

3	$3xy, 6yz, 9y^2$
y	$xy, 2yz, 3y^2$
	$x, 2z, 3y$

HCF = 3y

Check:

$3y(x + 2z - 3y) = 3xy + 6yz - 9y^2$

Exercise 6.2B

Factorize the following algebraic expressions.

a) $2x + 8y$

b) $6d + 12$

c) $6uv - 3v$

d) $6m^2 - 4mn$

e) $10x^2y^2 - 5xy$

f) $36fg + 27g^2$

g) $3a - 6b + 9c$

h) $u^2 - uv + uw$

Factorize $a^2 + 2ab + b^2$

To factorize $a^2 + 2ab + b^2$, we can write 2ab as ab + ab.

The HCF of a^2 and ab is a. Thus, $a^2 + ab = a(a + b)$.
The HCF of ab and b^2 is b. Thus, $ab + b^2 = b(a + b)$.

Hence, $a^2 + 2ab + b^2 = \underbrace{a^2 + ab} + \underbrace{ab + b^2}$

$= a(a + b) + b(a + b)$

$= (a + b)(a + b)$

$= (a + b)^2$

(a + b) is the common factor.

Thus, $a^2 + 2ab + b^2 = (a + b)(a + b)$

$= (a + b)^2$

Example 10

Factorize the following algebraic expressions.

a) $x^2 + 8x + 16$ b) $y^2 + 6yz + 9z^2$

Solution:

The HCF of x^2 and $4x$ is x. So $x^2 + 4x = x(x + 4)$.
The HCF of $4x$ and 16 is 4. So $4x + 16 = 4(x + 4)$.

a) $x^2 + 8x + 16 = x^2 + 4x + 4x + 16$

$= x(x + 4) + 4(x + 4)$

$= (x + 4)(x + 4)$

$= (x + 4)^2$

Check:

$(x + 4)^2 = (x)^2 + 2(x)(4) + (4)^2$
$= x^2 + 8x + 16$

TIPS

- $8x = 4x + 4x$
- $6yz = 3yz + 3yz$

b) $y^2 + 6yz + 9z^2 = y^2 + 3yz + 3yz + 9z^2$

$= y(y + 3z) + 3z(y + 3z)$

$= (y + 3z)(y + 3z)$

$= (y + 3z)^2$

Check:

$(y + 3z)^2 = (y)^2 + 2(y)(3z) + (3z)^2$

$= y^2 + 6yz + 9z^2$

Exercise 6.2C

Factorize the following algebraic expressions.

1. $k^2 + 8k + 16$
2. $h^2 + 6h + 9$
3. $b^2 + 12b + 36$
4. $u^2 + 10uv + 25v^2$
5. $16y^2 + 8yz + z^2$
6. $4t^2 + 12t + 9$

Factorize ab + ac + bd + cd

Example 11

Factorize the following algebraic expressions.

a) ax + 3a + 2x + 6

b) ab + by + ax + xy

TIPS

- Group two terms in the expressions as one group.
- Find the HCF of each group.
- Divide each group by the HCF to get the final quotient.
- Make sure the two final quotients are the common factors.

Solution:

a) $ax + 3a + 2x + 6$

$= (ax + 3a) + (2x + 6)$ ← Find the HCF of the expressions.

$= a(\underline{x + 3}) + 2(\underline{x + 3})$

Common factor

$= (x + 3)(a + 2)$

b) $ab + by + ax + xy$

$= (ab + by) + (ax + xy)$ ← Find the HCF of the expressions.

$= b(\underline{a + y}) + x(\underline{a + y})$

Common factor

$= (a + y)(b + x)$

Exercise 6.2D

Factorize the following algebraic expressions.

1. $4ac + 2ab + 2cd + bd$

2. $st + 4tv + 3su + 12uv$

3. $2ab + 2ac + b + c$

4. $4x^2 - 2xy - 6xz + 3yz$

Factorize $a^2 - b^2$

Since factorization is the reverse process of expansion, $a^2 - b^2$ can be factorized as $(a + b)(a - b)$.

$a^2 - b^2$ is also known as the difference of two squares.

$$a^2 - b^2 = (a + b)(a - b)$$

Example 12

Factorize the following algebraic expressions.

a) $4x^2 - y^2$

b) $8p^2 - 18$

Solution:

a) $4x^2 - y^2$

$= 2^2x^2 - y^2$

$= (2x)^2 - y^2$

$= (2x + y)(2x - y)$

b) $8p^2 - 18$

$= 2(4p^2 - 9)$

$= 2[(2p)^2 - 3^2]$

$= 2(2p + 3)(2p - 3)$

Exercise 6.2E

Factorize the following algebraic expressions.

1. $p^2 - q^2$
2. $a^2 - 3^2$
3. $x^2 - 9$
4. $1 - 9t^2$
5. $16p^2 - 25q^2$
6. $81u^2 - 100$
7. $2x^2 - 18y^2$
8. $18gh^2 - 50g$

Factorize and simplify algebraic fractions

An algebraic fraction is one in which the numerator or the denominator or both are algebraic expressions. Examples:

$$\frac{3a^2}{6}, \frac{8}{4ef}, \frac{12gh^2k}{3gh}, \frac{9p + 6pq}{6 + 4q}, \frac{x^2 - y^2}{x^2 + xy}$$

Example 13

Simplify the following expressions.

a) $\dfrac{4ab^2}{6a^2c}$ b) $\dfrac{6pq}{9p^2qr}$

Solution:

a) $\dfrac{4ab^2}{6a^2c} = \dfrac{2a(2b)}{2a(3ac)}$

- The common factor of 4ab2 and 6a2c is 2a.
- Divide both numerator and denominator by 2a.

$= \dfrac{2b^2}{3ac}$

No more common factor

b) $\dfrac{6pq}{9p^2qr} = \dfrac{3pq(2)}{3pq(3pr)}$

$= \dfrac{2}{3pr}$

- The common factor of 6pq and $9p^2qr$ is 3pq.
- Divide both numerator and denominator by 3pq.

No more common factor

Example 14

Simplify the following expressions.

a) $\frac{2x + 10}{xy + 5y}$

b) $\frac{m^2 - 9}{mn - 3n}$

Solution:

a) $\frac{2x + 10}{xy + 5y}$

$= \frac{2(x + 5)}{y(x + 5)}$

- Factorize both numerator and denominator.
- Divide by the common factor (x + 5).

$= \frac{2}{y}$

b) $\frac{m^2 - 9}{mn - 3n}$

$= \frac{(m + 3)(m - 3)}{n(m - 3)}$

- Factorize both numerator and denominator.
- Divide by the common factor (m - 3).

$= \frac{m + 3}{n}$

Exercise 6.2F

1. Simplify the following algebraic fractions.

a) $\frac{5ab}{10bc}$

b) $\frac{6p}{18pr}$

c) $\frac{9pq}{n}$

d) $\frac{fg}{4fgh}$

e) $\frac{w^2u}{wu}$

f) $\frac{12b^2c^2}{abc}$

2. Simplify the following algebraic fractions to the lowest terms.

a) $\dfrac{12m + 3}{6p}$

b) $\dfrac{4a^2 - 12ab}{6ab}$

c) $\dfrac{pq}{pr - ps}$

d) $\dfrac{u + v}{5u + 5v}$

e) $\dfrac{s - t}{s^2 - t^2}$

f) $\dfrac{f^2 - 9}{f^2 - 3f}$

6.3 ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS

Add or subtract two algebraic fractions with the same denominator

TIPS

- Add or subtract the numerators.
- Retain the denominators.
- + (a + b) = +a + b

+ (a – b) = +a – b

– (a + b) = - a – b

– (a – b) = - a + b

Example 15

Simplify the following.

a) $\frac{m + 3}{4} + \frac{2m + 1}{4}$ b) $\frac{k}{h + k} - \frac{k - 1}{h + k}$

Solution:

a) $\frac{m + 3}{4} + \frac{2m + 1}{4} = \frac{(m + 3) + (2m + 1)}{4}$

$= \frac{m + 3 + 2m + 1}{4}$

$= \frac{3m + 4}{4}$

b) $\dfrac{k}{h+k} - \dfrac{k-1}{h+k} = \dfrac{k-(k-1)}{h+k}$

$= \dfrac{k-k+1}{h+k}$

$= \dfrac{1}{h+k}$

Exercise 6.3A

Simplify the following to the lowest terms.

1. $\frac{a}{3} + \frac{b}{3}$

2. $\frac{9x}{11} - \frac{2x}{11}$

3. $\frac{mn}{8} + \frac{3mn}{8}$

4. $\frac{10}{d^2} + \frac{3}{d^2}$

5. $\frac{3a + 1}{4b^2} + \frac{a + 2}{4b^2}$

6. $\frac{7w}{15wuv} - \frac{4w}{15wuv}$

7. $\frac{2p + 5}{3q} + \frac{p - 2}{3q}$

8. $\frac{b + 1}{2b + 1} + \frac{b + 1}{2b + 1}$

Add or subtract two algebraic fractions with one denominator as a multiple of the other denominator

Example 16

Simplify the following to the lowest terms.

a)

Exercise 6.3B

Simplify the following to the lowest terms.

1. $\frac{a}{3} + \frac{a}{9}$

2. $\frac{2p}{5} - \frac{4p}{15}$

3. $\frac{y}{2} + \frac{y + 2}{6}$

4. $\frac{b}{st} + \frac{d}{rst}$

5. $\frac{2k}{m + 2} + \frac{4k}{3(m + 2)}$

6. $\frac{f + g}{hk} - \frac{g - f}{2hk}$

Exercise 6.3C

1. Simplify each of the following.

a) $\frac{a}{2} + \frac{a}{5}$

b) $\frac{h}{m} - \frac{3}{n}$

c) $\frac{b}{a} + \frac{a}{b^2}$

d) $\frac{1}{2b} - \frac{b}{ac}$

e) $\frac{3y}{4b} + \frac{5q}{7}$

f) $\frac{3d}{a} - \frac{2a}{3d}$

2. Simplify each of the following algebraic fractions.

a) $\frac{x}{6} + \frac{y}{9}$

b) $\frac{p}{6} - \frac{q}{8}$

c) $\frac{h}{2k} - \frac{h}{k}$

d) $\frac{a}{rt} + \frac{b}{st}$

e) $\frac{b + d}{p} - \frac{c}{pq}$

f) $\frac{w}{ab} + \frac{w + u}{b}$

6.4 MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS

Multiplication of two algebraic fractions

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

$$= \frac{ac}{bd}$$

Example 19

Exercise 6.4A

Find the product of the following.

1. $\frac{p}{3} \times 5q$
2. $(w + u) \times \frac{x}{y}$
3. $\frac{2t}{v^2} \times \frac{w}{u^2}$
4. $\frac{5}{6g} \times (-h)$
5. $(x + y) \times \frac{z}{x^2 + 1}$
6. $\frac{4}{p - q} \times \frac{p + q}{5}$

Exercise 6.4B

Find the quotient of the following.

1. $\frac{m}{3} \div 5n$

2. $\frac{m}{p} \div \frac{n}{qr}$

3. $\frac{b^2c}{d} \div \frac{a}{3c}$

4. $\frac{5}{6w} \div (-u)$

5. $(x + y) \div \frac{z + 1}{w}$

6. $\frac{s}{p - q} \div \frac{a}{b + d}$

Exercise 6.4C

1. Find the product of the following.

a) $\frac{p}{q} \times 3pq$

b) $15xy \times \frac{4}{20x}$

c) $\frac{uv}{wx} \times \frac{x^2}{u^2}$

d) $\frac{p}{qr} \times (-qr^2)$

e) $-\frac{rs^2}{4tu} \times \frac{uv}{r^2s}$

f) $\frac{4}{p-q} \times \frac{p^2 - q^2}{5}$

2. Divide the following.

a) $\frac{m}{3} \div mn$

b) $xyz \div \frac{wx}{y}$

c) $\frac{u^2}{6w} \div (-u)$

d) $(ax + ay) \div \frac{x + y}{w}$

e) $\frac{s}{p^2 - q^2} \div \frac{a}{p - q}$

f) $\frac{a + b}{c - d} \div \frac{a^2 - b^2}{3(c - d)}$

SUMMARY

ALGEBRAIC EXPRESSIONS III

Expansion

- The product of an algebraic term and an algebraic expression $a(b + c) = ab + ac$
- The product of two algebraic expressions $(a + b)(c + d)$
  $= ac + ad + bc + cd$
  $(a + b)^2 = a^2 + 2ab + b^2$
  $(a – b)^2 = a^2 – 2ab + b^2$
  $(a + b)(a – b) = a^2 – b^2$

Factorization

- Factorize an algebraic term $2ab = 2 \times a \times b$
- Factorize algebraic expressions
  $ab + ac = a(b + c)$
  $a^2 - b^2 = (a + b)(a – b)$
  $a^2 + 2ab + b^2 = (a + b)^2$
  $a^2 – 2ab + b^2 = (a – b)^2$
  $ab + ac + bd + cd = (b + c)(a + d)$

Algebraic fractions

Both or either the numerator or the denominator is an algebraic term or algebraic expression.

$$\frac{b}{5}, \frac{3}{a}, \frac{a+b}{a}, \frac{b}{a+b}, \text{ and } \frac{c-d}{a+b}$$